Chiral symmetry breaking with non-SUSY D7-branes in ISD backgrounds

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ABSTRACT: Supersymmetric probe branes satisfy the κ-symmetry condition which ensures that the action is minimized with respect to both variations of the embedding and of the world-volume gauge field. We observe that the κ-symmetry condition for a $D7$-brane in an imaginary self-dual (ISD) background can be generalized yielding minimization of the action with respect to variations of the gauge field only but not of the embedding. This provides a new way to construct non-BPS solutions for D7-branes once the embedding extremizes the geometrical volume of the brane. We then apply this method to the Klebanov-Strassler background and find a new $D7 - \bar{D}7$ brane configuration that realizes the spontaneous breaking of flavor chiral symmetry as evidenced by the Goldstone boson identified in the spectrum. This result generalizes our previous construction for the Klebanov-Witten model. We compare our setup with the Sakai-Sugimoto model and discuss possible applications to QCD-like physics.

Contents

1 Introduction

The steady progress of the gauge/string duality [1] in describing four-dimensional phenomena gives rise to a hope to find a stringy (gravity) model for the QCD and hadron physics. Yet among other unresolved problems finding a suitable model to capture the spontaneous breaking of flavor chiral symmetry (χSB) proven to be a complicated task. The fundamental matter appears as a result of incorporating the flavor branes [2] into the gravity background. To find such a solution is a formidable task. The problem greatly simplifies in the so-called quenched approximation $N_c \gg N_f$ when one can neglect the back-reaction of flavor branes and consider them in probe approximation. Even in this case to find a solution could be complicated, especially if a non-trivial background NS-NS flux is present.

The study of the flavor chiral symmetry breaking took a new turn after Sakai and Sugimoto proposed a model [3] providing a successful holographic description of the phenomenon. The Sakai-Sugimoto model is based on incorporating $D8$-branes into the background of non-extremal D4-branes. The former is also known as Witten's model [4]. The D8-branes have the U-like shape which can be interpreted as the stacks of D8 and anti D8-branes merging in the IR region. Since the branes and the anti-branes are separated in the UV, the flavor symmetry is $U(N_f)_L \times U(N_f)_R$. In the IR region though the flavor symmetry is broken to the diagonal $U(N_f)_D$ by a quark VEV. Besides the chiral symmetry breaking the Sakai-Sugimoto setup provides a holographic realization of light mesons and baryons.

Despite the success in describing chiral symmetry breaking the Sakai-Sugimoto model suffers from the drawbacks inherited from the Witten's model. In particular it is inconsistent in the UV region because the string coupling diverges there. Besides that the geometry includes a compact circle. Therefore the four dimensional description breaks down for the energies exceeding the compactification scale. A potential way to bypass these problems is to study the flavor branes in the confining backgrounds with constant dilaton like the one found by Klebanov and Stassler [5]. Motivated by the work of Sakai and Sugimoto we expect that to model the chiral symmetry breaking the flavor branes have to be U-like shaped with each end preserving a distinct set of supercharges while the whole solution is not supersymmetric. One way to achieve this in the KS case is to wrap the D7-brane over the three-cycle S^3 at the base of the conifold $T^{1,1} \cong S^2 \times S^3$ [6, 7] while D7-brane is also stretching in the rest of the confiold $R^+ \times S^2$ forming a U-like shape.

An important step in this direction was recently undertaken in [8] where the proposed $D7$-$\bar{D}7$ setup was investigated in the case of the conformal Klebanov-Witten (KW) background [6]. It was found there that the asymptotic location of two branches are not antipodal on the 2-sphere but have a fixed angle difference of $\sqrt{6}/4\pi$. Because the KW background is conformal the angle separation is independent of the only free parameter of the solution r_0, which marks the lowest point of the D7-brane's profile. In the dual field theory r_0 corresponds to a chiral symmetry breaking VEV. The corresponding Goldstone boson of the broken flavor chiral symmetry was also identified in the spectrum [8].

In an earlier attempt [9] a similar configuration of $D7 - \bar{D}7$ flavor branes was investigated in the confining background of the KS solution. Despite some interesting insights into the geometry of the D7-brane profile the solution was not completed because of complications due to the nontrivial background NS-NS flux of the Klebanov-Stassler (KS) solution. The main goal of this paper is to generalize the results of [8] to the Klebanov-Strassler background thus completing and generalizing the construction of [9]. As we have already mentioned above the main obstacle in this case is the non-trivial background B-field which couples to the word-volume fields and excludes the configurations with vanishing gauge field. Similarly to [8] the configuration is question should break supersymmetry. This and the non-linear nature of the probe's equation of motion significantly complicate the process of finding the solution.

In the quest for a probe brane solutions in the KS model (or any other background) a remarkable simplification can be achieved if one focuses only on the supersymmetric configurations. In this case a non-linear second order equations of motion can be substituted by the first order κ-symmetry constraint [10] which ensures that the D7's action is minimized. Thus in the case of the ISD background with constant dilaton the κ-symmetry requires the D7-brane to be embedded along a holomorphic cycle and the gauge invaraint world-volume gauge field $\mathcal{F} \equiv \varphi^\star(B) + 2\pi\alpha' F$ to be of the $(1,1)$ type and anti-self dual (ASD):

$$\mathcal{F} = - \star_4 \mathcal{F} \, . \tag{1.1}$$

Although to satisfy these requirements may turn to be a complicated task [11] in general the supersymmetric solutions are easier to find. Unfortunately the known supersymmetric solutions [12–14] have a geometry of the single branch which looks topologically trivial in

the UV and hence don't describe flavor chiral symmetry breaking in the dual gauge theory. Rather the flavor symmetry consists of a single $U(N_f)$. The holomorphic embddings has been also extensively studied beyond the quenched approximation taking into account the backreaction of the branes [15].

Our primary observation, however, is that the ASD condition (1.1) minimizes the D7-brane action with respect to the world-volume gauge field even though the other two criteria may not be met. Therefore it can be used to construct a non-supersymmetric non-BPS solutions provided one can solve the equation for the embedding. In fact the task is even simpler since the ASD gauge field decouples from the equations governing the embedding leading to an intuitive problem of extremizing geometrical volume of the D7-brane. In this paper we use this method to find a solution with the SU(2) symmetry which unambiguously fixes the embedding upon the boundary conditions in the UV are specified. The profile we found is not holomorphic, yet it minimizes the volume within its class of symmetries. We also complete the solution by constructing the ASD gauge field that satisfies the Bianchi identity.

The geometry of the solution is a reminiscent of the one found in the KW case [8]. It is a U-shaped configuration which wraps S^3 at the base of the conifold and is stretched along the equator on S^2. The only free parameter of the solution τ_0 measures the distance between the lowest point of the profile and the tip of the conifold. For τ_0 much larger than the deformation parameter of the conifold the geometry approaches the conformal one and the angle separation between the D7-brane and the anti-brane approaches the KW value $\sqrt{6}/4\pi$. For the configuration stretching all way to the tip $\tau_0 = 0$ the locations of brane and the anti-brane on S^2 are exactly antipodal.

The paper is organized as follows. In the next section we briefly review the κ-symmetry condition for the D7-brane and show that the ASD condition along minimizes the action with respect to the perturbations of the world-volume gauge field. We then proceed with formulating a method to find non-supersymmetric solutions via extremizing the geometrical volume of the embedding. In section 3 we construct such an embedding for the D7-brane in the KS background and accompany it by an appropriate ASD world-volume flux in section 4. In this way we construct the model with the spontaniously broken flavor chiral symmetry. We elaborate on the four dimensional physics of the model in section 5 and briefly discuss other possible applications of our findings in section 6. Various technical details are delegated to the appendices.

2   General idea

Motivated by the Introduction we are interested in finding a solution for a D7-brane embedded in a type IIB background with a Poincare symmetry and a constant dilaton $e^\phi = g_s$. As discussed in [16] such a background has imaginary-self-dual 3-form flux $G_3 = F_3 + \frac{i}{g_s}H_3$ and is usually referred as an ISD background. The ten dimensional geometry of an ISD background is a warped product

$$\mathrm{d}s^2_{(10)} = h^{-1/2}\left(\mathrm{d}x_0^2 + \ldots + \mathrm{d}x_3^2\right) + h^{1/2}\mathrm{d}s^2_{M_6}, \tag{2.1}$$

of the flat Minkowski space and a $6d$ dimensional CY space M_6 with the metric $\mathrm{d}s^2_{M_6}$. The warp factor h depends only on the coordinates along M_6 (radius τ in the conifold case) and determines the self-dual RR 5-form

$$\widetilde{F}_5 = (1 + \star_{10})\,\mathrm{d}C_4 \qquad \text{where} \qquad C_4 = h^{-1} \cdot \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_3. \tag{2.2}$$

The complex 3-form G_3 is zero in the Minkowski space and is imaginary self dual $G_3 = \star_6 i G_3$ with respect to the $6d$ metric on M_6. Providing that the 3-form is of the $(2, 1)$ type the background has four supercharges [16] (see [17] for the non-constant dilaton case) and can be dual to a certain $4d$ $\mathcal{N} = 1$ gauge theory.

In addition the self-duality of G_3 and the explicit form of C_4 in (2.2) imply that F_7 defined as

$$F_7 = \star_{10} F_3 + C_4 \wedge H_3 \qquad \text{with} \qquad \star_{10}\, F_3 = h^{-1} \star_6 F_3 \wedge \mathrm{d}x_0 \wedge \ldots \wedge \mathrm{d}x_3 \tag{2.3}$$

identically vanishes [13].

A particularly well known example of such a background is the Klebanov-Strassler solution [5]. Although the remnant of this section is applicable to any ISD background one can have KS background in mind in what follows. We particularly focus on the KS background starting section 3 where we construct a non-SUSY D7-brane illustrating the main idea of this section.

Now let us focus on a D7-brane that spans the Minkowski space together with a 4-cycle Σ_4 inside M_6. In order to find a classical D7-brane configuration we have to solve the equations of motion for both the scalars that parametrize the embedding and the gauge field living on the brane. The Poincare symmetry reduces the ten dimensional problem to six dimensions but the problem is still to complicated to be addressed in its generality. To find an appropriate embedding in M_6 is especially difficult if the background has a non-trivial NS-NS field $dB = H_3$. The situation, however, greatly simplifies if one considers a supersymmetric solution. The κ-symmetry analysis for the D7-brane reduces the problem to an Euclidean $D3$-brane extending along the 4-cycle Σ_4. The latter was analyzed in [10] where it was shown that in order to preserve the background supersymmetries the embedding and the world-volume gauge field have to satisfy the following conditions:

1. The cycle Σ_4 is holomorphic.

2. The gauge-invariant 2-form field strength

$$\mathcal{F} \equiv \varphi^\star(B) + 2\pi l_s^2 F \tag{2.4}$$

   is of type $(1, 1)$.

3. Field $\mathcal{F}$ is primitive, i.e. it is orthogonal to the pull-back of the Kähler form J

$$\varphi^*(J) \wedge \mathcal{F} = 0\;. \tag{2.5}$$

The last two requirements imply $\mathcal{F}$ being *anti-self dual*

$$\mathcal{F} = - \star_4 \mathcal{F}, \tag{2.6}$$

where the positive orientation on the 4-cycle Σ_4 is given by the volume form

$$w_C = \frac{1}{2} \varphi^\star (J \wedge J). \tag{2.7}$$

What happens, however, if we are after a *non-supersymmetric* D7-brane? Unlike the κ-symmetry condition above the anti-self duality condition (2.6) does not use the complex structure inherited from M_6 and it is natural to ask whether it can be relevant for a non-supersymmetric case. The answer can be formulated in the form of the following observation.

The classical equations of motion for the gauge fields on the D7-brane world-volume are solved by any anti-self dual field $\mathcal{F}$ (2.6). Furthermore, the anti-self dual gauge field $\mathcal{F}$ does not contribute to the equations of motion of the scalars that fix the embedding. In other words, the effective action for the scalars is the pure DBI action with zero gauge field i.e. the volume of the cycle Σ_4. Hence any embedding which extremizes the volume, together with any anti-self-dual gauge field on it solves the equations of motion for the probe D7-brane.

This is equally applicable to an anti-D7-brane if the anti-self-duality condition is substituted by self-duality.

Before we proceed with the proof let us notice that the observation above provides a novel way to construct classical solutions for D7-brane. At first one can find an embedding Σ_4 which extremizes the volume and later find gauge field configuration which is anti-self-dual and satisfies the Bianchi identity.

The observation directly follows from the fact that *for the given induced metric the D7-brane action is bounded from below and reaches its minimum if $\mathcal{F}$ is anti-self-dual.*

The rest of this section will prove the observation. The D7-brane action consists of the two parts

$$S_{D7} = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}, \tag{2.8}$$

where

$$S_{\mathrm{DBI}} = \mu_7 \int \mathrm{d}\sigma^8 e^{-\phi} \sqrt{- |\varphi^\star(g_{10}) + \mathcal{F}|} \qquad \text{and} \qquad S_{\mathrm{WZ}} = \mu_7 \int \sum_p C_p \wedge e^{\mathcal{F}}. \tag{2.9}$$

Here $\mu_7 = (2\pi)^7 l_s^8$, and $\varphi^\star(g_{10})$ is the pullback of the ten dimensional metric (2.1).

Let us now show that for the case at hand only the C_4 dependent Wess-Zumino term contributes to the action. First, we have shown above that $F_7 = 0$. Next, the C_2 term vanishes as we assume $\mathcal{F}$ has no no space-time legs due to Poincare symmetry. Finally, the term containing the Hodge dual of C_4 is excluded for the same reason as C_2.

Now we can plug the explicit form of C_4 (2.2) into the WZ action. Furthermore, since we are looking for a static classical solution, we can integrate over the space-time coordinates. The resulting action will depend only on the 4-cycle Σ_4 in M_6

$$S_{D7} = \mu_7 g_s^{-1} \mathrm{Vol}_{\mathbb{R}^{1,3}} \cdot \int_{\Sigma_4} \mathrm{d}\sigma^4 \left(\sqrt{|g + h^{-1/2} \mathcal{F}|} + \mathrm{Pf} \left(h^{-1/2} \mathcal{F} \right) \right). \tag{2.10}$$

Here g is the induced metric on the 4-cycle without the warp factor i.e. $\varphi^\star(g_{10}) = h^{-1/2}\mathrm{d}x_\mu\mathrm{d}x^\mu + h^{1/2}g$.

To find a classical solution we would have to consider the variation of this action with respect to the world-volume gauge fields A_μ and the scalars ϕ_i that determine the embedding. The former appear only in $\mathcal{F}$ but the latter in both g and $\mathcal{F}$. Remarkably, the analysis significantly simplifies if instead of writing down the explicit equations of motion we try to find the *global* minimum with respect to gauge field $\mathcal{F}$ for the given metric g. Such a minimum indeed exists due to an inequality

$$\sqrt{|\mathbf{1} + \mathcal{M}|} + \mathrm{Pf}\mathcal{M} \geqslant 1, \tag{2.11}$$

where $\mathcal{M}$ is some antisymmetric matrix. It is saturated when $\mathcal{M}$ is anti-self-dual. An easy way to analyze (2.10) is to choose the coordinates such that in a given point g is an identity matrix and therefore it is invariant under $so(4) \cong su(2) \times su(2)$ symmetry. At this point $\mathcal{M} = h^{-1/2}\mathcal{F}$ is antisymmetric and the action (2.10) can be written in the form (2.11). The antisymmetric $\mathcal{M}$ and can be represented as a sum of self-dual and anti-self-dual parts $\mathcal{M} = \mathcal{M}_s + \mathcal{M}_a$, each in a vector (adjoint) representation of one of the two $su(2)$'s. We denote the length of these two vectors as r_s and r_s. These lengths are the only invariants of $\mathcal{M}$ and hence the left-hand-side of (2.11) is a function of r_s and r_a. We further use $\mathrm{SU}(2) \times \mathrm{SU}(2)$ to align the "vectors" $\mathcal{M}_s$ and $\mathcal{M}_a$ along the third axis

$$\mathcal{M} = \begin{pmatrix} 0 & (r_s + r_a) & 0 & 0 \\ -(r_s + r_a) & 0 & 0 & 0 \\ 0 & 0 & 0 & (r_s - r_a) \\ 0 & 0 & -(r_s - r_a) & 0 \end{pmatrix}. \tag{2.12}$$

A straightforward calculation leads to inequality (2.11) written in the form

$$\sqrt{(1 + (r_a^2 - r_s^2))^2 + 4r_s^2} - (r_a^2 - r_s^2) \geqslant 1 \ . \tag{2.13}$$

Clearly this inequality is saturated for $r_s^2 = 0$ and arbitrary r_a^2 which means that $\mathcal{M}_s = 0$ and so $\mathcal{M}$ is anti-self-dual.

Now we see that the anti-self-dual field minimizes the action for any given g and the minimal value of the action is simply the geometrical volume. Therefore any embedding which extremizes the volume

$$S_{D7}^{\mathrm{eff}} = \mu_7 g_s^{-1} \mathrm{Vol}_{\mathbb{R}^{1,3}} \cdot \int_{\Sigma_4} \mathrm{d}\sigma^4 \sqrt{|g|}, \tag{2.14}$$

will solve the full set of the equations of motion if accompanied by any anti-self-dual field $\mathcal{F}$. If such a gauge field also satisfies the Bianchi identity

$$\mathrm{d}\mathcal{F} = \varphi^*(H_3), \tag{2.15}$$

the resulting configuration is a classical solution for the probe D7-brane.

Let us note here that the effective action (2.14) is calibrated by the integral of w_C (2.7) over Σ_4 and the calibration condition is satisfied when the embedding is holomorphic [18].

In this case ASD solution will satisfy the κ-symmetry condition and be supersymmetric. At the same time not all ASD solutions are supersymmetric. If embedding merely extremizes (but does not minimize) the volume, or has to satisfy certain boundary conditions incompatible with holomorphicity the resulting ADS solution is not supersymmetric. Similarly to [19] we will call such a solution non-BPS. It satisfies the first order constraint (2.6) and minimizes the action with respect to the perturbations of the world-volume gauge fields.

Moreover one can often easily investigate stability of the ASD solution. If an embedding (i.e. the resulting induced metric g) minimizes the volume (2.14) within a certain class of geometries (for example invariant under certain symmetry group), then an anti-self-dual solution "built" on top of this embedding is stable within this class of geometries as the action reaches its minimal value within the class. If there is more than one anti-self-dual gauge-field which satisfies Bianchi identity for the given metric g the solution might have zero modes associated with the unfixed D3-brane charge.

3 D7-branes in the KS case

To illustrate how the observation from section 2 can help one find new solutions we turn to the example of the KS background and construct an explicit example of a non-SUSY probe D7-brane. As was mentioned above the initial step in finding new solutions is to find an embedding Σ_4 which extremizes the volume. Certainly any holomorphic Σ_4 would do the job, but the resulting ASD solution would necessarily be supersymmetric. Alternatively one can focus on Σ_4 invariant under a sufficiently large group of symmetries which fixes the shape of Σ_4. Thus in the KS case one can focus on Σ_4 preserving a part of the global SO(4) symmetry of the conifold. Clearly any D7-brane necessarily breaks SO(4). The biggest unbroken subgroup a D7-brane can preserve is SU(2). There are two distinct ways how one can embed the unbroken SU(2) into SO(4) $\cong$ (SU(2)$_L$ × SU(2)$_R$) /$\mathbb{Z}_2$. The first one leaves the diagonal SU(2)$_D$ unbroken. The second one preserves only the SU(2)$_R$ (or equivalently the SU(2)$_L$) subgroup.

The SU(2)$_D$ invariant embedding leaves the homogeneous coordinate of the conifold z_4 invariant and acts on (z_1, z_2, z_3) as a vector. The lowest state in this sector is a supersymmetric embedding $z_4 = \text{const}$ discussed in [13]. As was mentioned in the Introduction this solution does not spontaneously break flavor chiral symmetry.

To find the solution preserving one of the two SU(2)'s we use the coordinates introduced in [20] which make it explicit. Following Papadopoulos and Tseytlin we represent the base of the conifold as the product $S^2 \times S^3$ with one of the SU(2) symmetries acting on the S^3 and leaving the S^2 invariant while the other SU(2) does not preserve the $S^2 \times S^3$ splitting and mixes the two. Let

$$e_1 = d\theta_1 \,, \quad e_2 = \sin\theta_1 d\phi_1 \tag{3.1}$$

be a basis of one-forms on S^2 and ϵ_i with $i = 1, 2, 3$ be a basis of the Maurer-Cartan SU(2)$_R$ invariant forms on S^3

$$d\epsilon_i = -\frac{1}{2}\epsilon_{ijk}\epsilon_j \wedge \epsilon_k \,. \tag{3.2}$$

Then the deformed conifold metric written in this notations is [20, 21]

$$ds^2_{M_6} = \frac{1}{4}\epsilon^{4/3}K(\tau)\cosh\tau\left[e_1^2 + e_2^2 + \epsilon_1^2 + \epsilon_2^2 + \right.$$
$$\left. + \frac{2}{\cosh\tau}(e_1\epsilon_1 + e_2\epsilon_2) + \frac{2}{3K(\tau)^3\cosh\tau}\left(d\tau^2 + \epsilon_3^2\right)\right],$$
$$\text{with}\quad K(\tau) = \frac{(\sinh\tau\cosh\tau - \tau)^{1/3}}{\sinh\tau}, \tag{3.3}$$

while the 10-dimensional metric appears in (2.1). Besides the metric we will also need the NS-NS form

$$B = h_1(\tau)\left[\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 + \frac{1}{\cosh\tau}\left(e_1 \wedge \epsilon_2 + \epsilon_1 \wedge e_2\right)\right],$$
$$\text{with}\quad h_1(\tau) = -\frac{1}{2}\left(g_s M l_s^2\right)\cdot\frac{\cosh\tau(\tau\cosh\tau - \sinh\tau)}{2\sinh^2\tau}. \tag{3.4}$$

The embedding of the D7-brane Σ_4 is stretching along the radius τ and three other direction on the base of the conifold. To be $SU(2)_R$ invariant Σ_4 must completely cover the three-sphere (which is maped into itself under the action of $SU(2)_R$) and be located at the distinct point(s) at S^2.

It is clear from (3.3) that the deformed conifold is not a direct product $R^+ \times S^2 \times S^3$. In principle it is possible to find a trivialization, i.e. an appropriate coordinate basis on S^3 independent of the position on S^2 [22]. This approach was adopted in [8] where a non-SUSY $SU(2)_R$ invariant D7-brane was found in the background of the singular KW solution. Here we generalize this configuration to the deformed conifold case but there is no apparent need to use the trivialization basis. In fact we are looking for the embedding covering S^3 completely and therefore the details of the coordinate basis on S^3 are not important. This would not be the case if we were after the non-$SU(2)_R$ invariant physics, such as the spectrum of meson perturbations for the model in question. As we will see below the induced metric on Σ_4 written in the ordinary coordinates splits into a direct product $R^+ \times S^3$ with the metric on S^3 being explicitly $SU(2)_R$ invariant.

As was outlined above we consider an embedding which covers S^3 and can be represented as a distinct point on S^2. The location of the latter may depend on the radius τ. Hence the embedding can be visualized as trajectory $(\phi_1(\tau), \theta_1(\tau))$ on S^2. The boundary condition at any given τ defines a plane which splits S^2 into two equal semi-spheres and preserves the reflection-symmetry through this plane. This implies that the trajectory $(\phi_1(\tau), \theta_1(\tau))$ is a part of a big circle and with help of the broken $SU(2)_L$ can be aliened along the equator $(\phi_1 = \phi(\theta), \theta_1 = \pi/2)$.[1] One can easily see that the resulting unwarped

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[1]It can be easily verified that choosing the longitude $(\phi_1 = \text{const}, \theta_1(\tau))$ would lead to exactly the same induced metric.

induced metric on Σ_4 is explicitly $SU(2)_R$ invariant (the dot stands for the $d/d\tau$ derivative)

$$ds^2_{\Sigma_4} = \frac{\epsilon^{4/3}}{4} K(\tau) \cosh\tau \left[\epsilon_1^2 + \tilde{\epsilon}_2^2 + \right.$$
$$\left. + \left(\tanh^2\tau \dot{\phi}^2 + \frac{2}{3K(\tau)^3 \cosh\tau} \right) d\tau^2 + \frac{2}{3K(\tau)^3 \cosh\tau} \epsilon_3^2 \right],$$

where $\quad \tilde{\epsilon}_2 \equiv \epsilon_2 + \dfrac{\dot{\phi}}{\cosh\tau} d\tau$. $\hspace{4cm}$ (3.5)

After integrating the directions along S^3 the effective action (2.14) becomes

$$S_{D7} = \mu_7 \frac{2\pi^2}{24} \epsilon^{8/3} \int d\tau \frac{\cosh(\tau)}{K(\tau)} \left(1 + \frac{3K(\tau)^3 \sinh^2(\tau)}{2\cosh(\tau)} \cdot \dot{\phi}^2 \right)^{1/2} , \hspace{1cm} (3.6)$$

which leads to

$$\dot{\phi}^2 = \frac{2\cosh\tau}{3K(\tau)^3 \sinh^2\tau} \left(\frac{K(\tau)\sinh^2\tau \cosh\tau}{K(\tau_0)\sinh^2\tau_0 \cosh\tau_0} - 1 \right)^{-1} . \hspace{1cm} (3.7)$$

This configuration does not necessarily stretches to the bottom of the conifold $\tau = 0$ but rather to the minimal radius $\tau = \tau_0$. There the value of $\dot{\phi}$ jumps from minus to plus infinity which means that the D7-brane turns back into the direction of larger radius. Hence the geometry is U-shaped, accompanied by a never-shrinking S^3 at each radius.

To calculate the total angle "traveled" by the trajectory on S^2 we integrate $\dot{\phi}$ from the lowest radius τ_0 to infinity

$$\frac{1}{2}\Delta\phi = \left(\frac{2}{3} \right)^{1/2} \int_{\tau_0}^\infty d\tau \left(\frac{\cosh\tau}{K(\tau)^3 \sinh^2(\tau)} \right)^{1/2} \left(\frac{K(\tau)\sinh^2(\tau)\cosh\tau}{K(\tau_0)\sinh^2(\tau_0)\cosh\tau_0} - 1 \right)^{-1/2} . \hspace{0.3cm} (3.8)$$

For a very large τ we have $K(\tau) \approx 2^{1/3} e^{-\tau/3}$ and one can check that for τ_0 large enough $\frac{1}{2}\Delta\phi = \frac{\sqrt{6}}{8}\pi$ which coincides with the KW case [8] . On the other hand for the small values of τ we have $K(\tau) \to (2/3)^{1/3}$ and so in the limit $\tau_0 \to 0$ the integral reduces to

$$\frac{1}{2}\Delta\phi = \lim_{\tau_0 \to 0} \int_{\tau_0}^\infty \frac{d\tau}{\tau \left(\left(\frac{\tau}{\tau_0} \right)^2 - 1 \right)^{1/2}} = \frac{\pi}{2}. \hspace{1cm} (3.9)$$

This means that for $\tau_0 = 0$ we have an antipodal configuration. figure 1 demonstrates the dependence of the angle $\Delta\phi$ on the parameter τ_0.

4 Anti-self-duality condition

Our nest task is to complete the embedding (3.7) by an anti self dual gauge field which satisfies Bianchi identity. Such a gauge field will minimize the action. Therefore it is reasonable to assume that it preserves the unbroken $SU(2)_R$ symmetry. Hence we proceed with an explicitly $SU(2)_R$ invariant ansatz for the gauge field on Σ_4 in the $A_\tau = 0$ gauge

$$A(\tau) = A_i(\tau)\epsilon_i . \hspace{1cm} (4.1)$$
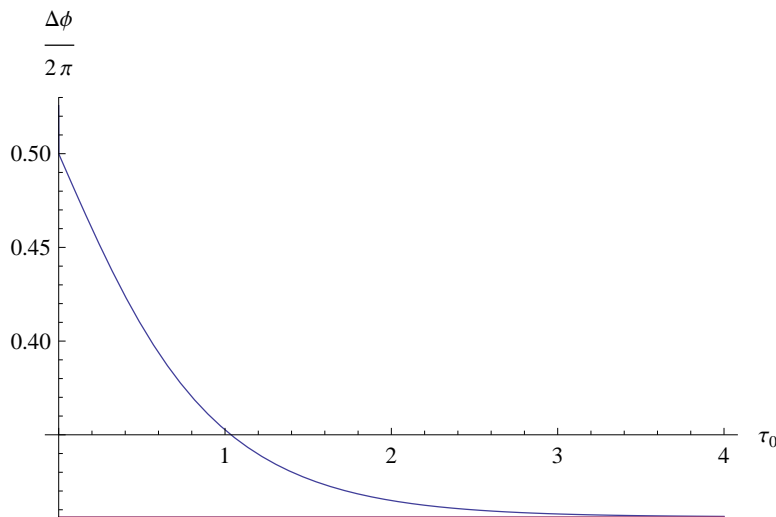
Figure 1. The blue plot shows the dependence of the angle $\Delta\phi$ in (3.8) on the lowest point of the profile τ_0. For $\tau_0 = 0$ the brane and the anti-brane are antipodal ($\Delta\phi = \pi$), whereas for $\tau_0 \to \infty$ we recover the KW result $\Delta\phi \to \sqrt{6}/4\pi$ (red line).

Given that the pull-back of the NS-NS form is

$$\varphi^*(B) = h_1(\tau)\epsilon_1 \wedge \tilde{\epsilon}_2\,, \tag{4.2}$$

and using (3.2) one can easily calculate the gauge-invariant combination (from here on we put $g_s M l_2^2 = 2$)

$$\mathcal{F} = \left(\dot{A}_3(\tau) + \frac{A_1(\tau)}{\cosh\tau}\dot{\phi}(\tau)\right) d\tau \wedge \epsilon_3 + \left(\dot{A}_1(\tau) - \frac{A_3(\tau)}{\cosh\tau}\dot{\phi}(\tau)\right) d\tau \wedge \epsilon_1 +$$

$$+ \dot{A}_2(\tau)d\tau \wedge \epsilon_2 - (A_3(\tau) - h_1(\tau))\epsilon_1 \wedge \tilde{\epsilon}_2 - A_1(\tau)\tilde{\epsilon}_2 \wedge \epsilon_3 + A_2(\tau)\epsilon_1 \wedge \epsilon_3. \tag{4.3}$$

Because of the U-like shape of the D7-brane's profile we can not use τ as a world-volume coordinate globally. Nevertheless we can start at one branch in the large τ region and write down down the anti-self-duality condition (2.6) using the induced metric (3.5)

$$\left(\dot{A}_1(\tau) - \frac{\dot{\phi}(\tau)}{\cosh\tau}A_3(\tau)\right) = L(\tau)A_1(\tau),$$

$$\dot{A}_2(\tau) = L(\tau)A_2(\tau),$$

$$\left(\dot{A}_3(\tau) + \frac{\dot{\phi}(\tau)}{\cosh\tau}A_1(\tau)\right) = \frac{2L(\tau)}{3K(\tau)^3\cosh\tau}(A_3(\tau) - h_1(\tau)), \tag{4.4}$$

where

$$L(\tau) = \left(1 + \frac{3K(\tau)^3\sinh^2\tau}{2\cosh\tau}\dot{\phi}(\tau)^2\right)^{1/2} = \left(1 - \frac{K(\tau_0)\sinh^2\tau_0\cosh\tau_0}{K(\tau)\sinh^2\tau\cosh\tau}\right)^{-1/2}. \tag{4.5}$$

– 10 –

These equations are valid everywhere on that branch from $\tau = \infty$ to $\tau = \tau_0$. When τ reaches τ_0 and we move to another branch the orientation flips (because $d\tau$ changes sign) and so $L \to -L$ in the equations above. To better understand this flip of sign we can switch from τ to the well-defined coordinate ϕ (3.7). Assuming now the gauge field (4.1) is ϕ-dependent $A = A(\phi)_i \epsilon_i$ we can write down the ASD equation valid everywhere on the profile. Let us put $A_1 = A_3 = 0$ and focus on A_2 for simplicity. The ASD equation in this case takes the form

$$\frac{\mathrm{d}A_2(\phi)}{\mathrm{d}\phi} = \tilde{L}(\phi)A_2(\phi) \tag{4.6}$$

$$\text{with} \quad \tilde{L} = \frac{L}{|\dot{\phi}|} = \left(\frac{3K(\tau)^3 \sinh^2 \tau}{2 \cosh \tau} \cdot \frac{K(\tau) \sinh^2 \tau \cosh \tau}{K(\tau_0) \sinh^2 \tau_0 \cosh \tau_0} \right)^{1/2} \quad \text{for} \quad \tau = \tau(\phi).$$

Since $L(\phi)$ and $\tilde{L}(\phi)$ are positively defined and the equation (4.6) is valid everywhere on D7-brane we have to conclude that the right-hand-side of the equations (4.4) changes sign when τ moves from one branch into another. The branch with the plus sign as in (4.4) describes the D7-brane while the other branch with minus sign in front of L is the anti-D7-brane.

Let us now analyze the ASD equations. We start with the D7-brane branch i.e. the equations (4.4). At large τ we have $L(\tau) \to 1$. The function $A_2(\tau)$ is decoupled from the other two functions and therefore it grows like e^τ if not zero. This means that we have to put $A_2(\tau) = 0$ to satisfy the boundary conditions at infinity.

We were not able to find an analytical solution for A_1 and A_3 for general τ_0. But for $\tau_0 = 0$ the derivative $\dot{\phi}(\tau)$ vanishes everywhere except the origin and the solution can be easily constructed. First we notice that in this case $A_1(\tau)$ satisfies the same equation as $A_2(\tau)$ in (4.4). Therefore the only non-divergent solution is $A_1(\tau) = 0$. A general solution for the remaining equation for $A_3(\tau)$ is

$$A_3(\tau) = e^{S(\tau)} \left(\int_\tau^\infty d\tau' \frac{2h_1(|\tau'|)}{3K(\tau')^3 \cosh \tau'} e^{-S(\tau')} + c_0 \right)$$

$$\text{with} \quad S(\tau) = \int_0^\tau d\tau' \frac{2}{3K(\tau')^3 \cosh \tau'} \tag{4.7}$$

and c_0 is an integration constant. It is easy to see that $S(\tau) \approx \frac{2}{3}\tau$ for large τ and so $A_3(\tau)$ exponentially diverges if c_0 is not zero. On the other hand, for $c_0 = 0$ we find for large τ

$$A_3(\tau) \to \frac{-\tau}{2} - \frac{1}{4}. \tag{4.8}$$

One can also check that $A_3(\tau)$ is regular at $\tau = 0$ for any value of c_0.

We analyze the equations for A_1 and A_3 for $\tau_0 > 0$ in appendix B and show that there is a unique solution regular at the UV and that it is regular at $\tau = \tau_0$.

To find the solution on the anti D7-brane branch we notice that the homogeneous solution to (4.4) with $-L$ instead of L is a decaying exponent at infinity. Therefore any solution is regular at UV. The solution is nevertheless uniquely fixed by the boundary condition at $\tau = \tau_0$ to ensure that the solutions on both branches are smoothly glued at
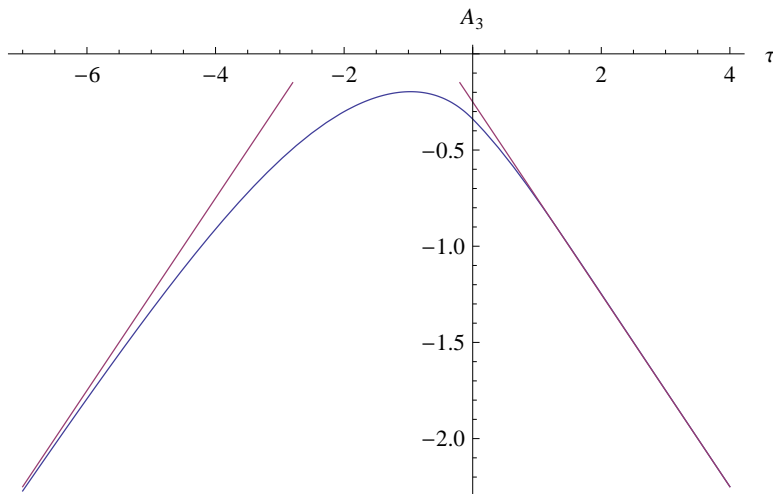
Figure 2. The blue plot is the numerical solution for $A_3(\tau)$ for $\tau_0 = 0$. The solution rapidly approaches its asymptotic behavior (4.8) and (4.9) (red line).

the junction. To prove that such a solution exists it is enough to switch to the coordinate ϕ like in (4.6) since the point $\tau = \tau_0$ is not special in this case.

To illustrate the point that the solution on both branches can be smoothly glued at $\tau_0 = 0$ we return to the function $A_3(\tau)$ in the special case $\tau_0 = 0$. In this case we can introduce negative τ to describe the anti D7 branch while the positive τ describes the D7 branch.[2] This choice is convenient because (4.7) continued to the negative values of τ is a solution on the $\bar{D}7$ branch.[3] The solution is obviously smooth at $\tau = \tau_0 = 0$ and rapidly approaches its asymptotic at plus (4.8) and minus infinity

$$A_3(\tau) \to \frac{-|\tau|}{2} + \frac{5}{4}, \tag{4.9}$$

as shown on figure 2.

To summarize, the unique choice of the non-divergent boundary conditions for the gauge fields $A_i(\tau)$ in the UV on the $D7$ branch unambiguously fixes the solution which is regular and smooth everywhere, including IR region $\tau = \tau_0$ and the UV region on the $\bar{D}7$ branch.

5 The dual gauge field theory

In [8] a solution for $N_{\rm f}$ probe D7 and anti D7 branes merging in the IR was found in the context of the conformal KW background. In the previous two sections we generalized

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[2] For $\tau_0 = 0$ we can define a new set of coordinates $y = \tau \cos(\phi)$ and $z = \tau \sin(\phi)$ which are similar to the coordinates introduced in [8] for the singular conifold. Now the $\tau_0 = 0$ embedding is simply given by $y = 0$ and z is the coordinate that spans the brane worldvolume. Moreover, z ranges from $-\infty$ to ∞ and the reflection $z \to -z$ exchanges the two branches of the brane profile like in [8]. The radial coordinate is given by $\tau = |z|$, where the negative values of τ now correspond to negative z. We therefore can rewrite the equation for A_3 in (4.4) entirely in terms of z while providing legitimacy to (4.7) for $\tau < 0$.

[3]Notice the modulus in the argument of h_1 in (4.7)

this result to the KS case. In this section we will address the phenomenological differences between the two models in question.

1. *Broken conformal symmetry*

   Because of the conformal symmetry of the KW background the asymptotic angle separation between the branes was "frozen" to $\Delta\phi = \frac{\sqrt{6}}{4}\pi$ and did not depend on the position of the lowest point along the profile r_0. Therefore the Goldstone boson $\frac{\partial}{\partial r_0}$ associated with the spontaneous symmetry breaking of scale invariance did not change boundary conditions i.e. was normalizable. As a result the massless Goldstone boson was present in the physical spectrum of the problem. This obviously is not the case in the KS model. The KS is a confining background and is obviously not scale invariant. As a result the angle separation $\Delta\phi$ varies between $\frac{\sqrt{6}}{4}\pi$ and π as one takes τ_0 from infinity to zero. Therefore the massless mode $\frac{\partial}{\partial\tau_0}$ changes the boundary conditions, namely not normalizable, and hence not in the spectrum.

2. *Broken chiral symmetry*

   As for chiral symmetry we expect that the scenarios in the KW and the KS are quite similar. In particular the underlying gauge theories both admit chiral symmetry which is spontaneously broken by a VEV. It was argued in [8] that the D7 branes described there give rise to the left and right *Weyl* spinors rather than the Dirac spinors. The argument was based on the observation that, if taken separately, each branch of the brane's profile is similar to one "half" of the holomorphic embedding of [12]. The latter setup includes Weyl spinors as can be shown using the "original" $\mathcal{N} = 2$ gauge theory [6]. Since in our case the branches are separated in the UV the Weyl spinors can not interact and hence massless. In the IR the chiral symmetry must be broken to the diagonal flavor symmetry due to a quark anti-quark condensate. The spontaneous breaking of the chiral symmetry gives rise to a Goldstone boson which can be identified as a massless mode (pion) in the mesonic spectrum on the gravity side. From the gauge theory perspective, one can argue that the chiral symmetry must be spontaneously broken based on the 't Hooft anomaly matching condition. This is similar to the situation in ordinary QCD and for $N_{\mathrm{f}} < N_c$ for $\mathcal{N} = 1$ SQCD.

   Let us show that the massless Goldstone mode is present in the spectrum also for the KS background. It was argued in [8] that the $5d$ Maxwell action takes the form (here the integral is symbolic and covers both branches)

$$S = -T' \int \mathrm{d}x^4 \mathrm{d}\tau \left(C(\tau) F_{\mu\nu} F^{\mu\nu} + 2D(\tau) F_{\mu\tau} F_\tau^{\ \mu} \right) \tag{5.1}$$

   and a zero mode in the vector spectrum appears if and only if the integral

$$\int_{\tau_0}^{\infty} \frac{\mathrm{d}\tau}{D(\tau)} \tag{5.2}$$

   is finite. We find the explicit form of the functions $C(\tau)$ and $D(\tau)$ in appendix C. It appears that $D(\tau)$ does not depend on the gauge fields on the S^3 and therefore is

the same on both branches

$$D(\tau) = |g|^{1/2} g^{\tau\tau} = \frac{\sqrt{3}}{4} \epsilon^{4/3} \frac{K(\tau) \cosh(\tau)}{L(\tau)} \ . \tag{5.3}$$

This expression is finite for $\tau \to \tau_0$ since $D(\tau_0)$ behaves like $(\tau - \tau_0)^{1/2}$ near $\tau_0 = 0$. Therefore the integral (5.2) is obviously convergent.

3. *Broken supersymmetry*

It was shown in [8] that the two branches of the D7-brane profile asymptotically can be described via

$$\frac{w_1}{w_3} = \lambda_1 \,, \qquad \frac{w_1}{w_3} = \lambda_2 \,, \qquad \frac{\lambda_1}{\lambda_2} = e^{i\Delta\phi} \ . \tag{5.4}$$

Here w_i are the homogeneous coordinates on the singular conifold

$$w_1 w_2 - w_3 w_4 = 0 \,, \tag{5.5}$$

and the two complex constants λ_1, λ_2 mark the location of the branches on S^2. Although each branch is asymptotically holomorphic together they do not lie on a holomorphic curve. Therefore the corresponding solution is not supersymmetric. Since the geometry of the deformed confiold approaches the singular conifold in the UV the description (5.4) is still valid in the KS case. Therefore the solution found in the previous sections is not supersymmetric. From the field theory perspective the broken supersymmetry implies that while the chiral quarks are massless, the corresponding flavor scalars (these would be squarks in a supersymmetric case) are massive. It also implies that the fermionic spectrum on the flavor brane should not admit a massless Goldstino. We expect this can be proved by a careful analysis of the D3-D7 strings.

Another important question that we would like to address is CP violation. We saw in the previous section that the ASD world-volume flux does not respect the $\mathbf{Z}_2$ symmetry that exchanges the two branches of the D7-brane (i.e. $\phi \to -\phi$). We argue that this $\mathbf{Z}_2$ breaking implies the C and P-symmetry breaking while CP is unbroken.

Before we turn to the analysis of the KS case let us review the results derived in [8] for D7-brane in the KW background. In this case the world-volume flux vanishes and the embedding is symmetric under $\mathbf{Z}_2$. It was shown there that both C and P transformations involve the $\mathbf{Z}_2$ flipping $\phi \to -\phi$. Yet one can assign charges to all fields such that the effective five and four-dimensional actions are C and P invariant. To illustrate the point we consider a five-dimensional interaction term

$$\int d^4x \ d\phi \ \delta\theta_1 \ F \wedge F \,, \tag{5.6}$$

which follows from the WZ term $\int_{D7} \tilde{C}_4 e^{\mathcal{F}}$. Here $\delta\theta_1$ is the KK mode of θ_1 defined in (3.1). Since C (and P) turn ϕ into $-\phi$ for all fields in (5.6) the expression is invariant no matter how the wave-functions F and $\delta\theta_1$ look like. The transformation $\phi \to -\phi$ is simply a

change of integration variable from the five-dimensional point of view. Furthermore since the embedding is $\mathbf{Z}_2$ invariant each individual world-volume wave-function has definite $\mathbf{Z}_2$ charge and therefore one can assign each fluctuation with a definite C (and P) charge. This is not so if there is a non $\mathbf{Z}_2$-invariant world-volume flux present, like in the case when D7 is embedded into KS background. It is still true that the interaction term (5.6) is invariant under C and P symmetry. But because the individual fluctuations satisfy a non-$\mathbf{Z}_2$ symmetric equation they don't carry a definite charge under $\mathbf{Z}_2$ and hence under C or P. Therefore both C and P symmetry is broken in four dimensions. Despite that the CP symmetry is not violated as the combination of C and P does not act on the fifth coordinate ϕ.

We leave it as an open problem to find an explicit equation for $\delta\theta_1$ (and other fields) and estimate the amount of C and P violation. In any case our scenario significantly differs from the Peccei-Quinn model where CP is broken spontaneously by a non-zero VEV.

Let us finish this section with a comment on possible applications of our result to the physics of baryons inspired by the Sakai-Sugimoto model. It was shown in [3] that the D8-brane warping the S^4-cycle gives rise to a five-dimensional gauge theory. The later has solutions with a non-zero baryon charge that correspond to the dissolved D4-branes warping S^4. On a more technical level the corresponding solutions are the five-dimensional instantones which do not shrink to the zero size due to a non-zero Chern-Simons term. Naturally our model has a potential for a similar construction. Indeed, in our case the D7-brane is wrapping the S^3-cycle and leads to a five-dimensional gauge theory if these directions are integrated out. We expect that a dissolved D3-brane that is wrapping S^3 will similarly give rise to a baryon configuration. We leave the detailed form of the five-dimensional action and other aspects of the baryon solution for future investigation.

6 Discussion

In this paper we proposed a new method to construct a non-BPS solutions for the probe D7-brane in the context of the type IIB backgrounds with a constant dilaton. Our method is based on the anti-self-duality condition for the world-volume gauge field which generalizes the κ-symmetry condition for the SUSY D-brane. We apply this method to the Klebanov-Strassler background and found a non-SUSY solution with a SU(2) symmetry. The resulting profile is a product of the Minkowsi space and a U-like shape accompanied at each point by a never-shrinking S^3. It can be thought of as a junction of a D7 and an anti D7-brane. Because of the merging region in the IR the asymptotic flavor symmetry $SU(N)_L \times SU(N)_R$ is spontaneously broken down to $SU(N)_D$ by a non-zero VEV. We identified the corresponding massless Goldstone boson in the physical spectrum. We expect our model to provide a holographic realization of the QCD-like physics (mesons and baryons) in a similar way to the Sakai-Sugimoto model. At the same time our model does not suffer from the divergent dilaton in the UV .

Besides describing flavor physics in the QCD-like theories the D7-brane plays a crucial role of stabilizing Kähler moduli in the throat compactification models proposed in [25]. The stringy models of inflation based on this setup include a mobile D3-brane [26] with its

dynamics governed by the D3-D7 interaction [27]. It was recently shown that such a model is capable of describing cosmological inflation matching current experimental date [28]. Given the multitude of possible locations of the D7-brane within the throat the dynamics of the model can significantly vary. The solution for the D7-brane found in this paper is rather peculiar in this sense as it preserves a large symmetry group and this significantly simplifies the analysis. Because of the $SU(2)_R$ invariance the only possible operators perturbing the effective potential of the inflaton are those with quantum numbers $j_2 = R = 0$ and arbitrary j_1. The analysis in [29] reveals that in this case instead of the generic form

$$V_{D3-D7}(r) \cong c_1 r + c_{3/2} r^{3/2} + c_2 r^2 + O(r^4) + \ldots , \quad c_i \sim 1 , \qquad (6.1)$$

the D3-D7 potential acquires a specific form

$$V_{D3-D7}(r) \cong c_2' r^2 + O(r^4) + \ldots , \qquad (6.2)$$

with a possibility to fine tune the mass parameter c_2' to zero. It would be interesting to study this model in more detail to investigate if it yields more favorable phenomenological predictions.

In conclusion let us note that the method of constructing non-supersymmetric non-BPS solutions discussed in section 2 can be applied to the interesting problem of holographic description of the SUSY breaking states in supersymmetric field theories. To construct such a state one has to find an embedding for the D7-brane which satisfies some "holomorphic" boundary condition in UV and solves the EOM for the effective action (2.14) but is not holomorphic. If this solution happens to be metastable it can be used in various applications including the holographic gauge mediation models like the one discussed in [30].

A The minimum of the D7-brane action in the case of a general metric

In this appendix we analyze the action (2.10) ans show that it is bounded from below by (2.14). The inequality is saturated if $\mathcal{F}$ is ASD.

Let $\mathcal{M}$ be a 4×4 matrix defined by[4] $\mathcal{M} = g^{-1}\mathcal{F}$, where g and $\mathcal{F}$ are arbitrary real symmetric and anti-symmetric 4×4 matrices respectively and additionally g is *positive definite*. Using this notations we can rewrite the essential part of (2.10) as

$$\sqrt{|g|}\left[\sqrt{|1 + \mathcal{M}|} - \sqrt{|\mathcal{M}|}\right] . \tag{A.1}$$

From the symmetry properties we immediately find that $\mathcal{M}^\dagger = -g\mathcal{M}g^{-1}$ and so the matrices $\mathcal{M}^\dagger$ and $(-\mathcal{M})$ are similar. As such they have the same eigenvalues and the same (block) diagonal form. Let us denote by D the (block) diagonalized form of $\mathcal{M}$ and by P a non-singular matrix satisfying $D = P\mathcal{M}P^{-1}$. We then find that $D^\dagger = -\left(P^{\dagger-1}gP^{-1}\right)D\left(Pg^{-1}P^\dagger\right)$. Since $D^\dagger$ and $(-D)$ have the same diagonal elements up to some re-ordering, the matrix g is necessarily *congruent* to a permutation matrix. The only permutation matrix, however, congruent to a *positive definite* matrix is the unit matrix. We conclude therefore that the eigenvalues of D, and $\mathcal{M}$, are purely imaginary. Notice, however, that $\mathcal{M}$ is by definition real and so if $i\lambda$ is an eigenvalue then also is $-i\lambda$. To summarize, the four eigenvalues of $\mathcal{M}$ are $\pm i\lambda_1$ and $\pm i\lambda_2$ for real $\lambda_{1,2}$. As a consistency check we can verify that $\mathcal{M}$ is indeed traceless as it follows from its definition. (Noticing further that $(-\mathcal{M})$ and $\mathcal{M}^T$ are also similar we can easily show that $\mathcal{M}$ is actually diagonalizable and not block-diagonalizable, but this is not necessary for our discussion).

Using these results we can write:

$$\sqrt{|1 + \mathcal{M}|} - \sqrt{|\mathcal{M}|} = \sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)} - |\lambda_1 \lambda_2| \geqslant 1, \tag{A.2}$$

where the inequality is saturated *if only if* $\lambda_1 = \lambda_2$. In this case all the eigenvalues of $\mathcal{M}^2$ are equal and so $\mathcal{M}^2$ is proportional to the unity matrix:

$$\mathcal{M}^2 = -|\mathcal{M}|^{1/2} \cdot \mathbf{1}. \tag{A.3}$$

In terms of the original matrices g and $\mathcal{F}$ it implies:

$$\epsilon^{abcd}\left(\star_4 \mathcal{F}\right)_{ab}\mathcal{F}_{ce} = -2\text{Pf}(\mathcal{F}) \cdot \delta_e^d, \tag{A.4}$$

where $\star_4$ stands for the Hodge duality with respect to the 4-dimensional metric g. Comparing this with the identity:

$$\epsilon^{abcd}\mathcal{F}_{ab}\mathcal{F}_{ce} = 2\text{Pf}(\mathcal{F}) \cdot \delta_e^d, \tag{A.5}$$

which holds for any 4×4 anti-symmetric matrix $\mathcal{F}$, we immediately[5] arrive at the anti self-duality condition (2.6).

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[4]Here we ignore the warp function h which can be trivially absorbed in g or $\mathcal{F}$.

[5] Going from (A.4) to (A.5) we assumed that $|\mathcal{F}| \neq 0$. Otherwise for $|\mathcal{F}| = 0$ we have $|\mathcal{M}| = 0$ an so from (A.3) one gets $\mathcal{M}^2 = 0$. This in turn means that the matrix $\mathcal{F}g^{-1}\mathcal{F}$ is zero, which is possible only for $\mathcal{F} = 0$ and thus the anti-self duality condition still holds.

B The solution of the ASD condition (4.4) for $\tau_0 > 0$

Here we consider the solution of the system of equations (4.4) for $\tau_0 > 0$. In this case $\dot{\phi} \neq 0$ (or, more precisely, $\dot{\phi}(\tau)$ is not a δ-function) and we cannot put $A_1(\tau) = 0$ anymore. On the other hand we still have to put $A_2(\tau) = 0$ to avoid the divergence at infinity. If we denote the vector $(A_1(\tau), A_3(\tau))^T$ by $A(\tau)$ then the equations (4.4) can be written as:

$$\dot{A}(\tau) = Q(\tau)A(\tau) + R(\tau), \tag{B.1}$$

where the 2×2 matrix $Q(\tau)$ and the 2-vector $R(\tau)$ can be easily read from (4.4). The solution of this differential equation is:

$$A(\tau) = e^{S(\tau)} \left(\mathrm{P} \int_\tau^\infty e^{-S(\tau')} R(\tau') \mathrm{d}\tau' + C_0 \right). \tag{B.2}$$

Here $S(\tau)$ is defined by $S'(\tau) = Q(\tau)$ and the integration constant can be absorbed in a proper redefinition of the constant vector C_0. The ordering in the solution appears because of the matrix nature of the equation. For $\tau \to \infty$ we have:

$$Q(\tau) \approx \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \qquad \text{and} \qquad R(\tau) \approx \begin{pmatrix} 0 \\ \frac{\tau}{3} \end{pmatrix}. \tag{B.3}$$

Thus if $C_0 \neq (0,0)^T$ we have $A_1(\tau) \approx e^\tau$ or $A_3(\tau) \approx e^{\frac{2}{3}\tau}$. Therefore we have to put $C_0 = (0,0)^T$ so that at infinity $A_1(\tau)$ vanishes and $A_3(\tau) \approx -\frac{\tau}{2}$. Now we have to verify that the solution in question is regular at $\tau = \tau_0$. We find near τ_0

$$Q(\tau) \approx \frac{1}{(\tau - \tau_0)^{1/2}} Q_0 \qquad \text{and} \qquad R(\tau) \approx \begin{pmatrix} 0 \\ \frac{1}{(\tau-\tau_0)^{1/2}} \end{pmatrix} R_0 . \tag{B.4}$$

for some non-zero R_0 and non-singular Q_0. This implies that for small $\tau - \tau_0$, $S(\tau) \approx (\tau - \tau_0)^{1/2} S_0$. Finally both $A_1(\tau)$ and $A_3(\tau)$ behave like $\mathrm{const} + (\tau - \tau_0)^{1/2}$ at $\tau = \tau_0$ and thus the solution is regular in both UV and the IR. Clearly this solution can be continued to the second branch. Since the solution to the homogeneous system behaves like $A_1(\tau) \approx e^{-\tau}$ or $A_3(\tau) \approx e^{-\frac{2}{3}\tau}$ at infinity the solution in question is regular everywhere.

C The general form of the 5D Maxwell action

In this appendix we will derive the general form of the 5d effective action of the gauge fields.

To this end we have to expand the D7 brane action (2.9) around the classical profile. The perturbation we are interested in is $\delta\mathcal{F} = 2\pi l_s^2 F$ where F has legs only along the 4d space-time and the radial coordinate τ. As usual the contribution of the WZ part is proportional to $F \wedge F$ and does not contribute to the equations of motion for an abelian $\mathcal{F}$. After the integration over the 3-sphere the variation of the DBI part yields

$$\delta S_{\mathrm{DBI}} = -T' \int d^4x d\tau \sqrt{|E|} \mathrm{Tr}\left((E^{-1}F)^2 \right) \tag{C.1}$$

with $E \equiv \varphi^\star(g_{10}) + \mathcal{F}_0$, where $\varphi^\star(g_{10})$ the pullback of the 10d metric g_{10} (2.1) and $\mathcal{F}_0$ is the ASD 2-from we found in section 3. Here we absorbed various numerical and dimensionful constants inside T' and use the fact that $\mathrm{Tr}(E^{-1}F)$ vanishes. The latter follows from the fact that $\delta\mathcal{F}$ has legs only along the space-time while $\mathcal{F}_0$ has no such legs. The 8×8 matrix E consists of two 4×4 blocks. The first block corresponds to the 4d space time and the second is related to the 4-cycle of the deformed conifold warped by the D7 brane. The 8×8 matrix E can be written schematically as $E = (h^{-\frac{1}{2}}\eta_{\mu\nu}, h^{\frac{1}{2}}g(1+\mathcal{M}_0))$ where the entries correspond to the two blocks. As was already explained after equation (2.10) in our notations g is the un-warped induced metric on the 4-cycle, namely $\varphi^\star(g_{10}) = h^{-1/2}\mathrm{d}x_\mu\mathrm{d}x^\mu + h^{1/2}g$ and $\mathcal{M}_0 = h^{-1/2}g^{-1}\mathcal{F}_0$. In the same notation we have $F = (F_{\mu\nu}, F_{\mu\tau})$.

It was demonstrated in appendix A that the ASD classical solution we are investigating satisfies (A.3). This in turn implies that

$$(\mathbf{1} + \mathcal{M}_0)^{-1} = \frac{(\mathbf{1} - \mathcal{M}_0)}{\left(1 + \sqrt{|\mathcal{M}_0|}\right)}. \tag{C.2}$$

Using this result we find

$$E^{-1} = \left(h^{\frac{1}{2}}\eta^{\mu\nu}, \frac{h^{-\frac{1}{2}}\sqrt{|g|}}{\sqrt{|g|} + h^{-1}\sqrt{|\mathcal{F}_0|}}\left(g^{-1} - h^{-\frac{1}{2}}g^{-1}\mathcal{F}_0 g^{-1}\right)\right). \tag{C.3}$$

and

$$\sqrt{|E|} = \sqrt{|g|} + h^{-1}\sqrt{|\mathcal{F}_0|}. \tag{C.4}$$

Finally, noticing that $\left(g^{-1}\mathcal{F}_0 g^{-1}\right)^{\tau\tau} = 0$, we arrive at the following 5d Maxwell action

$$S_{5d} = -T'\int d^4x d\tau \left[\left(h\sqrt{|g|} + \sqrt{|\mathcal{F}_0|}\right)F_{\mu\nu}F^{\mu\nu} + 2\sqrt{|g|}g^{\tau\tau}F_{\mu\tau}F^\mu_\tau\right]. \tag{C.5}$$

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